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OPERATOR THEORETICAL REALIZATION OF SOME GEOMETRIC NOTIONS

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ABSTRACT. This paper studies the realization of certain geometric constructions in Cowen-Douglas operator class. Through this realization, some operator theoretical phenomena are easily seen from the corresponding geometric phenomena. In particular, we use this technique to solve the first-order equivalence problem and introduce a new operation among certain operators.

The nature of Cowen-Douglas theory is to identify operators of a certain type with certain geometric objects.

Based on this idea, we work on certain geometric constructions, holomorphic curves in $\mathrm{Gr}(n, \mathbf{C}^{2n})$ (the Grassmannian of n-dim subspaces of $\mathbf{C}^{2n}$) in Part 1 and tensor product of vector bundles in Part 2, and seek their operator theoretical realization.

Our realization of holomorphic curves in $\mathrm{Gr}(n, \mathbf{C}^{2n})$ will preserve important relations, and can be informally viewed as the imbedding of holomorphic curves in $\mathrm{Gr}(n, \mathbf{C}^{2n})$ into the Cowen-Douglas operator class $B_n(\Omega)$. Using this realization, we solve the first-order equivalence problem by explicitly exhibiting two operators $T_1, T_2 \in B_n(\mathcal{D})$ such that T_1 is not unitarily equivalent to T_2 but T_1 and T_2 have identical curvatures.

The realization of tensor product of vector bundles gives a natural operation among Cowen-Douglas operators. Using this operation, certain operator theoretical phenomena have been clarified naturally. E.g., for certain $g \in H^\infty$, the corresponding Bergman operator B_g^* is the "square" of the corresponding Toeplitz operator T_g^*.

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PART 1. OPERATOR THEORETICAL REALIZATION OF HOLOMORPHIC CURVES IN $\mathrm{Gr}(n, \mathbf{C}^{2n})$

1.1. Introduction. We will state only the main point of Cowen-Douglas theory here, and refer the reader to [**C-D, 1**] for further details.

If H is a separable Hilbert space, and Ω is an open connected subset of $\mathbf{C}$, then the operator class $B_n(\Omega)$ is by definition

$$\{T \in \mathcal{L}(H) : \begin{array}{ll} 1. & \mathrm{range}(T - w) = H, \text{ if } w \in \Omega; \\ 2. & \dim\ker(T - w) = n, \text{ if } w \in \Omega; \\ 3. & \bigvee_{w\in\Omega} \ker(T - w) = H \text{ (spanning property)}\}. \end{array}$$

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We call the operators in $B_n(\Omega)$ Cowen-Douglas operators.

The fundamental relation between $T \in B_n(\Omega)$ and the associated n-dim holomorphic Hermitian vector bundle over Ω defined by

$$E_T\colon \ker(T - w) \\ \downarrow \\ w$$

is the following identification:

THEOREM A [**C-D, 1**]. *Two operators T and $\tilde{T}$ in $B_n(\Omega)$ are unitarily equivalent $\Leftrightarrow$ E_T and $E_{\tilde{T}}$ are equivalent as holomorphic Hermitian vector bundles. We write $E_T \cong E_{\tilde{T}}$.*

On the other hand, the *Calabi Rigidity Theorem* gives a perfect identification of a holomorphic curve (with spanning property)[1] $\gamma\colon \Omega \to \mathrm{Gr}(n, \mathbf{C}^{2n})$ with its pull-back of the universal subbundle (i.e. $\gamma^*(S(n, \mathbf{C}^{2n}))$).

Our philosophy is they are related through their identified vector bundles.

DEFINITION 1.1.1. Let $F = (f_{ij})_{i,j}$ be a $n \times n$ matrix with H^∞ entries and let $g \in H^\infty$; we define the operator $S(g, F)$ by

$$S(g, F) \stackrel{\text{def}}{=} (T^*_{g\otimes I_n} \oplus T^*_{g\otimes I_n})|_{\mathrm{Graph}(T^*_F)}$$

where T^*_F and $T^*_{g\otimes I_n}$ are matrix Toeplitz operators acting on row vectors in $H^2 \otimes \mathbf{C}^n$.

It turns out that $S(g, F)$ is the right operator realization of the holomorphic curve, $\mathrm{span}\binom{I}{F}\colon \mathcal{D} \to \mathrm{Gr}(n, \mathbf{C}^{2n})$, where $\mathcal{D}$ is the open unit disk.

The geometric nature of $S(g, F)$ will be discussed in §1.2 and its operator theoretical nature will be discussed in §1.4.

(In this paper, the geometric part of a Cowen-Douglas operator or of a holomorphic curve means the geometric part of its *corresponding vector bundle.*)

1.2. Geometric aspects of this realization. In this section, we will show that for certain $g \in H^\infty$, the associated operator $S(g, F)$ belongs to $B_n(\Omega)$, and that through this realization, i.e. from the pull-back of the universal subbundle by $\mathrm{span}\binom{I}{F}\colon \mathcal{D} \to \mathrm{Gr}(n, \mathbf{C}^{2n})$ to $E_{S(g,F)}$, the important geometric relations are preserved.

In order to do this, we need to recall the definitions of some geometric invariants.

Let E, $\tilde{E}$ denote two n-dim Hermitian holomorphic vector bundles over an open connected set $\Omega \subset \mathbf{C}$. Let D_E denote the canonical connection of E and $D_E^2 = K_E\, dz\, d\bar{z}$ be its curvature tensor. We sometimes write K_E as K when no confusion arises.

It is well known that K is a C^∞ selfadjoint bundle map of E to E.

DEFINITION 1.2.1. If $\phi\colon E \to E$ is a C^∞ bundle map, then define ϕ_z and $\phi_{\bar{z}}$ by

$$[D_E, \phi] = D_E\phi - \phi D_E = \phi_z dz + \phi_{\bar{z}}\, d\bar{z};$$

$[D_E, \phi]$ is a C^∞ bundle map of E to $E \otimes \mathcal{E}'(\Omega)$, where $\mathcal{E}'(\Omega)$ denotes the set of C^∞ 1-forms over Ω.

[1] A holomorphic curve γ in $\mathrm{Gr}(n, \mathbf{C}^{2n})$ has the spanning property if $\sum_{z\in\Omega} \gamma(z) = \mathbf{C}^{2n}$. In this paper, we only consider holomorphic curves with this propety.

Here ϕ_z, $\phi_{\bar{z}}$ are clearly bundle maps of $E \to E$; they are called the first covariant derivatives for ϕ.

Taking the first covariant derivatives of ϕ_z, $\phi_{\bar{z}}$ and taking covariant derivatives of their covariant derivatives, etc., we get higher order covariant derivatives of ϕ.

For details see [**C-D, 2**].

REMARK. Relative to any C^∞ frame S of E, write $\Gamma(S)\,dz + \tilde{\Gamma}(S)\,d\bar{z}$ for the connection 1-form matrix of D_E and $\phi(S)$ for the matrix representation of ϕ relative to S. Then

$$\phi_z(S) = [\Gamma(S), \phi(S)] + \partial\phi(S)/\partial z, \quad \phi_{\bar{z}}(S) = [\tilde{\Gamma}(S), \phi(S)] + \partial\phi(S)/\partial\bar{z}.$$

The covariant derivatives of the curvature bundle map K_E give the important geometric invariants of E.

DEFINITION 1.2.2. Let E, $\tilde{E}$ be n-dim Hermitian holomorphic vector bundles and let k be a positive integer. We say E is equivalent to order k with $\tilde{E}$, if for each $z \in \Omega$, there is an isometry $\phi_z \colon E_z \to \tilde{E}_z$ such that $\phi_z \circ \chi = \tilde{\chi} \circ \phi_z$, where χ is a covariant derivative of K with total order $\leq k$, but bi-order $(p, q) \neq (0, k)$ or $(k, 0)$, and $\tilde{\chi}$ is the corresponding covariant derivative for $\tilde{K}$. (We shall say χ has total order $\leq k$ and satisfies the bi-order condition.)

For example, E is equivalent to order 1 with $\tilde{E} \Leftrightarrow$ for each $z \in \Omega$, there is an isometry $\phi_z \colon E_z \to \tilde{E}_z$ such that $\phi_z \circ K = \tilde{K} \circ \phi_z$. (We say E and $\tilde{E}$ have identical curvatures.)

THEOREM B [**C-D, 2**]. *If* $\dim E = \dim \tilde{E} = n$, *then* $E \cong \tilde{E} \Leftrightarrow E$ *and* $\tilde{E}$ *are equivalent to order* n.

We list two simple facts related to Definition 1.2.2:

(1) If $\tilde{\Omega}, \Omega \subset \mathbf{C}$, $g \colon \tilde{\Omega} \to \Omega$ is an analytic function, then E_1 and E_2 are equivalent to order k, so are $g^*(E_1)$ and $g^*(E_2)$.

(2) If E_1 and $\tilde{E}_1$, E_2 and $\tilde{E}_2$ are both equivalent to order k respectively, so are $E_1 \otimes E_2$ and $\tilde{E}_1 \otimes \tilde{E}_2$.

For an explanation of this, see [**L**].

If T_1 and T_2 are in $B_n(\Omega)$, the relation of E_{T_1} and E_{T_2} being equivalent to order k is directly reflected in the relation of T_1 and T_2.

THEOREM C [**C-D, 1**]. *If* $T_1, T_2 \in B_n(\Omega)$, *then* E_{T_1} *and* E_{T_2} *are equivalent to order* $k \Leftrightarrow T_1|_{\ker(T_1 - w)^{k+1}}$ *and* $T_2|_{\ker(T_2 - w)^{k+1}}$ *are unitarily equivalent for each* $w \in \Omega$.

In this situation, we will say T_1 and T_2 are equivalent to order k.

NOTATION. We will use $\bar{\Omega}$ to denote the conjugate of a subset Ω of $\mathbf{C}$ and $\mathrm{bd}(\mathcal{D})$ to denote the boundary of $\mathcal{D}$.

The following lemma is a characterization of $T_g^* \in B_1(\Omega)$ for $g \in H^\infty$.

LEMMA 1.2.1. *If* $g \in H^\infty$, Ω *connected open in* $\mathbf{C}$, *then* $T_g^* \in B_1(\Omega) \Leftrightarrow$ *the map* $g \colon g^{-1}(\bar{\Omega}) \to \bar{\Omega}$ *is onto and is a conformal equivalence.*

PROOF. Recall

$$(*) \qquad (T_g^* - \overline{g(\bar{z})})k_z = 0,$$

where $z \in \mathcal{D}$ and $k_z(\varsigma) = 1/(1 - z\varsigma)$ for $\varsigma \in \mathcal{D}$.

" $\Rightarrow$ " The mapping $g\colon g^{-1}(\bar{\Omega}) \to \bar{\Omega}$ has to be injective, because $z_1 \neq z_2$ in $\mathcal{D}$ implies k_{z_1} and k_{z_2} are linearly independent.

The fact that $g = g^{-1}(\bar{\Omega}) \to \bar{\Omega}$ is surjective follows from:

(1) $\bar{\Omega} \subset \overline{\sigma(T_g^*)} = \sigma(T_g) = \operatorname{clos}(g(\mathcal{D}))$;

(2) $\bar{\Omega} \cap \sigma_e(T_g) = \overline{\Omega \cap \sigma_e(T_g^*)}$ is empty;

(3) $\sigma_e(T_g) \supset \operatorname{bd}(g(\mathcal{D}))$. (See [**D**].)

" $\Leftarrow$ " *Step* 1. Fix any $w = g(z_0) \in \bar{\Omega}$, then $g(z) - w = (z - z_0)h(z)$.

We claim h is invertible in H^∞.

It is trivial to see $h \in H^\infty$ and h is nowhere zero in $\mathcal{D}$.

The invertibility of h in H^∞ follows from observing that for any $z_n \in \mathcal{D}$, with $g(z_n) \to w$ (assume $g(z_n) \in \bar{\Omega}$), we have $g^{-1}(g(z_n)) = z_n \to z_0$ (because $g\colon g^{-1}(\bar{\Omega}) \to \bar{\Omega}$ is a conformal equivalence).

Step 2. Let $w = g(z_0) \in \bar{\Omega}$ as above. Notice that $\ker(T_g - w) = 0$.

We claim $\operatorname{range}(T_g - w) = \{f \in H^2\colon f(z_0) = 0\}$.

This follows from two facts:

(1) $[(T_g - w)(f)](z) = (g(z) - w)f(z) = (z - z_0)h(z)f(z)$;

(2) if $f(z) = (z - z_0)l(z)$ with $l \in H^2$, then

$$f(z) = [(z - z_0)h(z)](l(z)/h(z)) = [(T_g - w)(l/h)](z).$$

Thus $f - (f(z_0)/k_{\bar{z}_0}(z_0))k_{\bar{z}_0} \in \operatorname{range}(T_g - w)$. Thus $T_g - w$ has closed range and using formula $(*)$, we have

$$\operatorname{span}(k_{\bar{z}_0}) \oplus \operatorname{range}(T_g - w) = H^2.$$

So $\dim(\ker(T_g^* - \bar{w})) = \dim[\operatorname{coker}(T_g - w)] = 1$, and $T_g^* - \bar{w}$ is Fredholm of index 1.

This shows $T_g^* \in B_1(\Omega)$. □

From now on, E_F will denote the holomorphic Hermitian vector bundle

$$\begin{array}{c} \operatorname{span}\begin{pmatrix} I \\ F(z) \end{pmatrix}, \\ \downarrow \\ z \end{array}$$

where $F = (f_{ij})_{i,j}$ is an $n \times n$ matrix of analytic functions on $\Omega \subset \mathbf{C}$; $\binom{I}{F(z)}$ is always viewed as a collection of column vectors in $\mathbf{C}^{2n}$.

THEOREM 1.2.2. *If $g \in H^\infty$ and $T_g^* \in B_1(\Omega)$, then $S(g, F) \in B_n(\Omega)$ for any $F = \{f_{ij}\}_{i,j}$ with each $f_{ij} \in H^\infty$. Moreover*

1. *$E_{S(g,F)} \cong E_{T_g^*} \otimes E_{\mathcal{F}}$, where $\mathcal{F}(z) = \overline{F(g^{-1}(\bar{z}))}$;*

2. *E_F and E_G are equivalent to order $k \Leftrightarrow S(g, F)$ and $S(g, G)$ are equivalent to order k (where G has entries in H^∞).*

PROOF. It is trivial to see that $S(g, F) \in B_n(\Omega)$, since $B_n(\Omega)$ is closed under similarity and $S(g, F) \sim T^*_{g\otimes I_n}$ (by the graph mapping $x \mapsto (x, T_F^* x)$). We go directly to 1 and 2.

1. Observe that $k_{\overline{g^{-1}(\bar z)}}$ is a holomorphic frame of $E_{T_g^*}(k_w(\varsigma) = 1/(1-\varsigma w))$, that $(k_{\overline{g^{-1}(\bar z)}} \otimes I_n) \oplus \mathcal{F}(z)(k_{\overline{g^{-1}(\bar z)}} \otimes I_n)$ is a holomorphic frame of $E_{S(g,F)}$, and that $\binom{I}{\mathcal{F}(z)}$ is a holomorphic frame of $E_{\mathcal{F}}$. Therefore

$$(k_{\overline{g^{-1}(\bar z)}} \otimes I_n) \oplus \mathcal{F}(z)(k_{\overline{g^{-1}(\bar z)}} \otimes I_n) \to k_{\overline{g^{-1}(\bar z)}} \otimes \begin{pmatrix} I \\ \mathcal{F}(z) \end{pmatrix}$$

is the desired holomorphic isometric bundle map. (In the expression above, both sides are thought of as collections of n-vectors.)

2. Using the fact (2) following Definition 1.2.2, we see $E_{\mathcal{F}}$ is equivalent to order k with $E_{\mathcal{G}} \Leftrightarrow E_{S(g,F)}$ and $E_{S(g,G)}$ are equivalent to order k, where $\mathcal{G}(z) = \overline{G(g^{-1}(\bar z))}$.

Once we prove "E_F and E_G are equivalent to order $k \Leftrightarrow E_{\mathcal{F}}$ and $E_{\mathcal{G}}$ are equivalent to order k for $g(z) \equiv z$," then the rest follows directly from fact (1).

NOTE. Over the holomorphic frame $\binom{I}{F(z)}$, the connection 1-form matrix of E_F is $\{(I+F^*(z)F(z))^{-1}F^*(z)F'(z)\}\, dz$ and so the matrix of its curvature bundle map K_{E_F} is

$$-(I+F^*(z)F(z))^{-1}(F'(z))^*(I+F(z)F^*(z))^{-1}F'(z) \quad (\text{see } [\mathbf{C\text{-}D}, \mathbf{1}]).$$

So by the remark following Definition 1.2.1, over the holomorphic frame $\binom{I}{F(z)}$ the matrix representations of the covariant derivatives of K_{E_F} on E_F are all noncommutative polynomials in $F^{(i)}(z)$, $\overline{F^{(j)}(z)}$ and $(I+F^*(z)F(z))^{-1}$, $(I+F(z)F^*(z))^{-1}$, $i,j \geq 0$. Also such polynomials are canonical in the sense that they are independent of the choice of F. So for a covariant derivative of E_F, the conjugate of its matrix (relative to the frame $\binom{I}{F}$) at $\bar z$ is exactly the corresponding one for $E_{\mathcal{F}}$ (relative to the frame $\binom{I}{\mathcal{F}}$) at z.

From Definition 1.2.2, the rest of the proof is quite straightforward. □

Notice that E_F is the pull-back of the universal bundle under the holomorphic mapping $z \to \operatorname{span}\binom{I}{F(z)} \in \operatorname{Gr}(n, \mathbf{C}^{2n})$; in view of the Calabi Rigidity Theorem, Theorem 1.2.2 above says the geometry of these realization operators mirrors the geometry of holomorphic curves in $\operatorname{Gr}(n, \mathbf{C}^{2n})$.

COROLLARY 1.2.3. *If there is a z_0 in $\mathcal{D}$ (unit disk) with $F(z_0) = G(z_0) = 0$, then $S(g,F) \cong S(g,G) \Leftrightarrow$ there are constant unitary matrices V, W such that $VF(z)W \equiv G(z)$ on $\mathcal{D}$.*

PROOF. From Theorems A, B and 1.2.2, this corollary really says that $E_F \cong E_G \Leftrightarrow \exists$ constant unitary matrices V, W such that $VF(z)W \equiv G(z)$.

By the Calabi Rigidity Theorem, $E_F \cong E_G \Leftrightarrow \exists$ constant $2n \times 2n$ unitary matrix

$$\mathcal{U} = \begin{pmatrix} U_1 & U_2 \\ U_3 & U_4 \end{pmatrix},$$

where each U_j is an $n \times n$ matrix, such that

$$\begin{pmatrix} U_1 & U_2 \\ U_3 & U_4 \end{pmatrix} \begin{pmatrix} I \\ F(z) \end{pmatrix} \equiv \begin{pmatrix} I \\ G(z) \end{pmatrix} A(z),$$

where $A(z)$ is an $n \times n$ invertible matrix for each $z \in \mathcal{D}$.

" $\Rightarrow$ " We have the identity

$$(I,0)\mathcal{U}^*\mathcal{U} \begin{pmatrix} I \\ F(z) \end{pmatrix} = A^*(z_0)(I,0) \begin{pmatrix} I \\ G(z) \end{pmatrix} A(z),$$

if $z \in \mathcal{D}$ which implies $I \equiv A^*(z_0)A(z)$ and therefore $A(z) \equiv A(z_0)$ is unitary.

But $U_1 + U_2F(z) \equiv A(z_0)$ and $F(z_0) = 0$ implies $A(z_0) = U_1$ is unitary, hence $U_2 = U_3 = 0$ and U_4 is unitary.

So $U_4F(z) \equiv G(z)U_1$ in $\mathcal{D}$.

"$\Leftarrow$" $\begin{pmatrix} W^* & 0 \\ 0 & V \end{pmatrix}\begin{pmatrix} I \\ F(z) \end{pmatrix} \equiv \begin{pmatrix} I \\ G(z) \end{pmatrix} W^*$ on $\mathcal{D}$ gives $E_F \cong E_G$. □

1.3. The first-order equivalence problem. We seek two operators $T_1, T_2 \in B_n(\Omega)$ such that $T_1 \not\cong T_2$ but $T_1|_{\ker(T_1-w)^2} \cong T_2|_{\ker(T_2-w)^2}$ for each $w \in \Omega$.

Using Theorem C and Theorem 1.2.2, this problem is reduced to a geometric problem on $\mathrm{Gr}(n, \mathbf{C}^{2n})$, namely "Find two holomorphic curves f_1, f_2 in $\mathrm{Gr}(n, \mathbf{C}^{2n})$ such that $f_1^*(S(n, \mathbf{C}^{2n}))$ and $f_2^*(S(n, \mathbf{C}^{2n}))$ have the same curvature, but are inequivalent." Recall first that the Calabi Rigidity Theorem says $f_1^*(S(n, \mathbf{C}^{2n})) \cong f_2^*(S(n, \mathbf{C}^{2n})) \Leftrightarrow f_1$ and f_2 are identical up to a unitary action of $\mathbf{C}^{2n}$.

Second, fix an orthonormal basis of $\mathbf{C}^{2n}$, say $e_1, \dots, e_{2n}$; then $((e_1, \dots, e_{2n})X, (e_1, \dots, e_{2n})Y) \to Y^{\mathrm{T}}X$ is a nondegenerated bilinear form. It is not hard to see that it induces an automorphism of $\mathrm{Gr}(n, \mathbf{C}^{2n})$. Call this kind of automorphism a correlation of $\mathrm{Gr}(n, \mathbf{C}^{2n})$.

Recall the Plücker imbedding of $\mathrm{Gr}(n, \mathbf{C}^{2n}) \to \mathbf{P}(\bigwedge^n \mathbf{C}^{2n})$ is the mapping $\mathrm{span}\{Z_1, \dots, Z_n\} \to \mathrm{span}(Z_1 \wedge Z_2 \wedge \cdots \wedge Z_n)$.

If $\mathbf{P}(\bigwedge^n \mathbf{C}^{2n})$ carries the Fubini-Study metric, then the canonical Kähler structure of $\mathrm{Gr}(n, \mathbf{C}^{2n})$ is induced by this holomorphic imbedding. (See [**Chern**].)

With this metric on $\mathrm{Gr}(n, \mathbf{C}^{2n})$, every correlation of $\mathrm{Gr}(n, \mathbf{C}^{2n})$ is an isometry and is in fact the unique nontrivial isometric automorphism of $\mathrm{Gr}(n, \mathbf{C}^{2n})$ up to the action of $U(2n)$ on $\mathrm{Gr}(n, \mathbf{C}^{2n})$. (See [**Chow**], [**Cowen**].)

Fix a correlation composed with a unitary $\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ action:

$$\phi\colon \begin{pmatrix} I \\ F \end{pmatrix} \to \begin{pmatrix} -F^{\mathrm{T}} \\ I \end{pmatrix} \to \begin{pmatrix} I \\ F^{\mathrm{T}} \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\begin{pmatrix} -F^{\mathrm{T}} \\ I \end{pmatrix},$$

where F is an $n \times n$ matrix, I is the identity $n \times n$ matrix and F^{T} is the transpose of F. We shall show that for any holomorphic curve $f\colon \Omega \to \mathrm{Gr}(n, \mathbf{C}^{2n})$, $f^*(S(n, \mathbf{C}^{2n}))$ and $(\phi \circ f)^*(S(n, \mathbf{C}^{2n}))$ have the same curvature, but there is an f such that we cannot get $\phi \circ f$ by any unitary action on f.

LEMMA 1.3.1. *The vector bundles E_1, E_2 are equivalent to order one $\Leftrightarrow$ for any C^∞ frame S_j on E_j $(j = 1, 2)$, $K_1(S_1)$ is similar to $K_2(S_2)$ pointwise.*

PROOF. Notice that the matrix representation of curvature changes by similarity under change of frame and the curvature of the canonical connection is selfadjoint.

So E_1 is equivalent to order one with $E_2 \Leftrightarrow$ the eigenvalues of K_1 and K_2 are the same. □

In the following two lemmas, we write $F = (f_{ij})_{i,j}$, $\tilde{F} = (\tilde{f}_{ij})_{i,j}$, where all f_{ij}, $\tilde{f}_{ij}$ are analytic functions on $\Omega \subset \mathbf{C}$.

LEMMA 1.3.2. *E_F and $E_{F^{\mathrm{T}}}$ are equivalent to order one.*

PROOF. Over the holomorphic frame $\begin{bmatrix} I \\ F(z) \end{bmatrix}$, K_E has matrix

$$(**) \qquad -(I + F^*F)^{-1}(F')^*(I + FF^*)^{-1}F'.$$

Using the elementary fact if A, B are invertible matrices, then $AB \sim BA$ and $A \sim A^{\mathrm{T}}$, it is obvious that $(**)$ is similar to

$$\begin{aligned} &-\{F'(I+F^*F)^{-1}(F')^*(I+FF^*)\}^{\mathrm{T}} \\ &\quad = -(I+(F^{\mathrm{T}})^*F^{\mathrm{T}})^{-1}[(F^{\mathrm{T}})']^*(I+F^{\mathrm{T}}(F^{\mathrm{T}})^*)^{-1}(F^{\mathrm{T}})'. \quad \square \end{aligned}$$

LEMMA 1.3.3. *Fix $z_0 \in \Omega$ and suppose that $F(z_0) = \tilde{F}(z_0) = 0$, and*

$$F'(z_0) = \tilde{F}'(z_0) = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \quad \textit{with } |\lambda_i| \neq |\lambda_j| \ (\textit{if } i \neq j).$$

Then $E_F \cong E_{\tilde{F}}$ implies $|f_{ij}(z)| \equiv |\tilde{f}_{ij}(z)|$ for all i, j and $z \in \Omega$.

PROOF. By Corollary 1.2.3,

$$\begin{aligned} E_F \cong E_{\tilde{F}} &\Leftrightarrow \exists \text{ constant unitary } n \times n \text{ matrices} \\ &\qquad V,\ W \text{ such that } VF(z)W \cong \tilde{F}(z) \text{ on } \Omega \\ &\Rightarrow VF'(z_0)W = \tilde{F}'(z_0) \\ &\Rightarrow W^* \begin{pmatrix} |\lambda_1|^2 & & & \\ & |\lambda_2|^2 & & \\ & & \ddots & \\ & & & |\lambda_n|^2 \end{pmatrix} W \\ &= \begin{pmatrix} |\lambda_1|^2 & & & \\ & |\lambda_2|^2 & & \\ & & \ddots & \\ & & & |\lambda_n|^2 \end{pmatrix} \\ &= V \begin{pmatrix} |\lambda_1|^2 & & & \\ & |\lambda_2|^2 & & \\ & & \ddots & \\ & & & |\lambda_n|^2 \end{pmatrix} V^* \\ &\Rightarrow W,\ V \text{ are both diagonal.} \quad \square \end{aligned}$$

COROLLARY 1.3.4. *Take any $F = (f_{ij})_{i,j}$ with*

1. *$f_{ij} \in H^\infty$ and $f_{ij}(0) = 0$ for all i, j;*
2. $$F'(z_0) = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}, \qquad |\lambda_i| \neq |\lambda_j|, \ \textit{if } i \neq j;$$
3. *$|f_{ij}| \neq |f_{ji}|$ for some i, j.*

Then $E_F \not\cong E_{F^{\mathrm{T}}}$.

We can now summarize the solution of the first-order equivalence problem as follows.

THEOREM 1.3.5. *If $n > 1$, and F is as in Corollary 1.3.4, then $S(z,F)$, $S(z,F^{\mathrm{T}}) \in B_n(\mathcal{D})$, $S(z,F)$ and $S(z,F^{\mathrm{T}})$ are equivalent to order one, but they are not unitarily equivalent.*

1.4. The operator theoretical aspect of this realization. We begin with a powerful theorem of Brown-Douglas-Filmore [**BDF**].

THEOREM D [**BDF**]. *Two essentially normal operators T_1 and T_2 are unitarily equivalent modulo compact operators $\Leftrightarrow$ $\sigma_e(T_1) = \sigma_e(T_2) = X$ and $\operatorname{ind}(T_1 - \lambda) = \operatorname{ind}(T_2 - \lambda)$ whenever $\lambda \in \mathbf{C} - X$.*

Notice that similarity of the operators T_1 and T_2 already implies the conditions on the right of Theorem D.

LEMMA 1.4.1. *If T, S are two bounded linear operators on H such that T is essentially normal and $[T,S] = 0$, then*

1. $\operatorname{Graph}(S)$ *is invariant under* $T \oplus T$;
2. $T \oplus T|_{\operatorname{Graph}(S)} = T_s$ *is unitarily equivalent to a compact perturbation of* T.

PROOF. 1 is trivial. For 2, note $T_s \sim T$ via the map $\phi\colon x \to (x, Sx)$. In view of [**BDF**], it suffices to show $[T_s, T_s^*]$ is compact.

Let P be the orthogonal projection of $H \oplus H$ onto $\operatorname{Graph}(S)$, then if $x \in H$,

$$T_s^*(\phi(x)) = P(T^*x \oplus T^*Sx) = \phi(T^*x) + P(0 \oplus [T^*, S]x).$$

Define $K\colon \operatorname{Graph}(S) \to \operatorname{Graph}(S)$ by $K(\phi(x)) = P(0 \oplus [T^*, S]x)$.

Note that by Fuglede's theorem in the Calkin algebra (i.e., $ts = st$, $t^*t = tt^*$ " $\Rightarrow$ " $t^*s = st^*$) $[T^*, S]$ is compact, thus K is a compact operator. Then

$$T_s^*(\phi(x)) = \phi(T^*x) + K(\phi(x)),$$

and

$$T_s \circ T_s^*(\phi(x)) = \phi(T \circ T^*x) + T_s \circ K(\phi(x)),$$
$$T_s^* \circ T_s(\phi(x)) = T_s^*(\phi(Tx)) = \phi(T^* \circ Tx) + K \circ T_s(\phi(x)).$$

Thus $[T_s, T_s^*] = \phi \circ ([T, T^*]) \circ \phi^{-1} + [T_s, K]$. □

THEOREM 1.4.2. *If $g \in H^\infty \cap QC$, then $S(g,F) \cong (T_{g\otimes I_n}^* + K) \sim T_{g\otimes I_n}^*$, where K is a compact operator*

$$QC = \overline{(H^\infty + C(S'))} \cap (H^\infty + C(S')).$$

PROOF. Since g is quasi-continuous, $T_{g\otimes I_n}^*$ is essentially normal (see [**D**]); the lemma above can then be applied. □

We know very little about the compact operator K in Theorem 1.4.2. One situation in which we do have some information is that of the following theorem, here stated without proof.

THEOREM 1.4.3. *If $0 < |a| < 1$, then $S(z, (z-a)/(1-\bar{z}a)) \cong U_+^* + K$, where U_+ is the unilateral shift on the orthonormal basis $\{e_n\}_{n=0}^\infty$ and $K(e_j) = 0$, $j \geq 2$, $K(e_0) = \bar{a}e_0$, $K(e_1) = be_0$ with $2|1+b|^2 + |a|^2 = 1$.*

PART 2. OPERATOR THEORETICAL REALIZATION OF TENSOR PRODUCT OF VECTOR BUNDLES

We will use the definitions and notations introduced in Part 1. Besides, we shall use $\alpha = (\alpha^{(1)}, \alpha^{(2)}, \dots, \alpha^{(n)})$ to denote an ordered n-tuple of vectors in the Hilbert space H (i.e. $\alpha^{(j)} \in H$), and define $\alpha^*\beta = (\langle \beta^{(j)}, \alpha^{(i)} \rangle)_{i,j}$, the $n \times n$ Gramian matrix of α and β, and $||\alpha||^2 \overset{\text{def}}{=} \operatorname{tr}(\alpha^*\alpha) = \sum_{i=1}^n ||\alpha^{(i)}||^2$. We shall say $\alpha \perp \beta$, if $\alpha^*\beta = 0$.

Moreover, let $\alpha_j = (\alpha_j^{(1)}, \alpha_j^{(2)}, \dots, \alpha_j^{(n)})$ be a sequence of ordered n-tuple vectors, $j = 1, 2, \dots$. We shall say $\{a_j\}_{j=1}^\infty$ is a linearly independent set, if $\{\alpha_j^{(k)} : 1 \le k \le n,\ j = 1, 2, \dots\}$ is a linearly independent set in H. We write

$$\operatorname{span}\{\alpha_j\}_{j=1}^\infty \overset{\text{def}}{=} \operatorname{span}\{\alpha_j^{(k)} : 1 \le k \le n,\ j = 1, 2, \dots\}.$$

Notice that if γ_1 is an ordered m-tuple vector and γ_2 is an ordered n-tuple vector, then $\gamma_1 \otimes \gamma_2$ is an ordered mn-tuple vector.

2.1. The operator realization of tensor product of vector bundles. Theorem A in Part 1 says, for $T \in B_n(\Omega)$, T and E_T are identified. Now, it is natural to ask: if $T_1 \in B_m(\Omega)$ and $T_2 \in B_n(\Omega)$, is $E_{T_1} \otimes E_{T_2}$ identified with some operator in $B_{mn}(\Omega)$?

The answer is affirmative, and hence we get a natural operation: $B_m(\Omega) \times B_n(\Omega) \to B_{mn}(\Omega)$. We shall call this operation the "geometric tensor product."

DEFINITION 2.1.1. Let $T_1 \in B_m(\Omega)$, $T_2 \in B_n(\Omega)$, where T_j is defined on H_j (separable Hilbert space), $j = 1, 2$. We define the subspace $H(T_1) * H(T_2)$ of $H_1 \otimes H_2$ by

$$H(T_1) * H(T_2) = \operatorname*{span}_{z\in\Omega}[\ker(T_1 - z) \otimes \ker(T_2 - z)]$$

and the operator $T_1 * T_2$ by

$$T_1 * T_2 \overset{\text{def}}{=} (T_1 \otimes I)|_{H(T_1)*H(T_2)}.$$

Observe that $H(T_1) * H(T_2)$ is a common invariant subspace of $T_1 \otimes I$ and $I \otimes T_2$. Thus $T_1 * T_2$ is well defined. Moreover $(T_1 \otimes I)|_{H(T_1)*H(T_2)} = (I \otimes T_2)|_{H(T_1)*H(T_2)}$ and $||T_1 * T_2|| \le \min(||T_1||, ||T_2||)$.

LEMMA 2.1.1. *Let W be any open subset of Ω, then*

$$H(T_1) * H(T_2) = \operatorname*{span}_{z\in W}[\ker(T_1 - z) \otimes \ker(T_2 - z)],$$

where T_1, T_2 are as in Definition 2.1.1.

PROOF. Let γ_j be a global holomorphic frame of E_{T_j} (cf. [G]), $j = 1, 2$. By Definition 2.1.1,

$$H(T_1) * H(T_2) = \operatorname{span}\{\gamma_1(z) \otimes \gamma_2(z) : z \in \Omega\}.$$

Also,

$$\operatorname*{span}_{z\in W}[\ker(T_1 - z) \otimes \ker(T_2 - z)] = \operatorname{span}\{\gamma_1(z) \otimes \gamma_2(z) : z \in W\}.$$

The lemma then follows from the application of the Identity Theorem in complex analysis. □

THEOREM 2.1.2. *Let $T_1 \in B_m(\Omega)$ and $T_2 \in B_n(\Omega)$, then $T_1 * T_2 \in B_{mn}(\Omega)$ and $E_{T_1 * T_2} = E_{T_1} \otimes E_{T_2}$.*

PROOF. *Step* 1. We claim that for each $z \in \Omega$,

$$\ker(T_1 * T_2 - z) = \ker(T_1 - z) \otimes \ker(T_2 - z).$$

Fix $z \in \Omega$, since $(T_1 * T_2) - z = [(T_1 - z) \otimes I]|_{H(T_1)*H(T_2)}$, clearly $\ker[(T_1 * T_2) - z] \supset \ker(T_1 - z) \otimes \ker(T_2 - z)$.

Conversely, for any $x \in H_1 \otimes H_2$ write $x = x_1 + x_2 + y$, where

$$x_1 \in \ker(T_1 - z) \otimes \ker(T_2 - z), \qquad x_2 \in \ker(T_1 - z) \otimes [\ker(T_2 - z)]^\perp,$$
$$y \in [\ker(T_1 - z)]^\perp \otimes H_2.$$

Notice that since $(T_1 - z) \otimes I$ and $I \otimes (T_2 - z)$ are both onto, as linear mappings $[(T_1 - z) \otimes I]|_{[\ker(T_1 - z)]^\perp \otimes H_2}$ and $[I \otimes (T_2 - z)]|_{H_1 \otimes [\ker(T_2 - z)]^\perp}$ are both invertible.

Now, if $x \in \ker(T_1 * T_2 - z)$, then $[(T_1 - z) \otimes I]y = [(T_1 * T_2) - z]x = 0$, so $y = 0$; moreover $[I \otimes (T_2 - z)]x_2 = [(T_1 * T_2) - z]x = 0$, so $x_2 = 0$. Thus $x = x_1 \in \ker(T_1 - z) \otimes \ker(T_2 - z)$.

Step 2. We claim $T_1 * T_2 - z$ is onto for each $z \in \Omega$. (Notice that $T_1 * T_2 \in B_{mn}(\Omega)$ will then follow from Step 1 and Step 2.)

Since $T_1 * T_2 - z = (T_1 - z) * (T_2 - z)$ (by Definition 2.1.1), it will be enough if we assume $0 \in \Omega$ and get a right inverse of $T_1 * T_2$.

Let $S_j = T_j^*(T_j T_j^*)^{-1}$, $j = 1, 2$. It is easy to see (1) $T_j S_j = \mathrm{id}$ for $j = 1, 2$; (2) if e_j is an orthonormal basis of $\ker T_j$ ($j = 1, 2$), then $(1 - zS_j)^{-1} e_j$ is a local holomorphic frame of E_{T_j} near 0 ($j = 1, 2$) and $e_j \perp S_j^k e_j$ for all $k \geq 1$.

Let $[|z| < r] \subset \Omega$ such that the two series $\sum_{k=0}^\infty z^k (S_j^k e_j)$, $j = 1, 2$, both converge in this disk. Then by Lemma 2.1.1,

$$\operatorname*{span}_{0<|z|<r} \left\{ \left[\sum_{k=0}^\infty z^k (S_1^k e_1) \right] \otimes \left[\sum_{k=0}^\infty z^k (S_2^k e_2) \right] \right\} = H(T_1) * H(T_2).$$

If P is the orthogonal projection of H_1 onto $\ker T_1$, then the following computation shows $H(T_1) * H(T_2)$ is invariant under $(S_1 \otimes I) + (P \otimes S_2)$:

$$\begin{aligned}
&[(S_1 \otimes I) + (P \otimes S_2)] \left\{ \left(\sum_{k=0}^\infty z^k S_1^k e_1 \right) \otimes \left(\sum_{k=0}^\infty z^k S_2^k e_2 \right) \right\} \\
&\quad = \left[\sum_{k=0}^\infty z^k S_1^{k+1} e_1 \right] \otimes \left[\sum_{k=0}^\infty z^k S_2^k e_2 \right] + e_1 \otimes \left[\sum_{k=0}^\infty z^k S_2^{k+1} e_2 \right] \\
&\quad = \frac{1}{z} \left\{ \left[\sum_{k=1}^\infty z^k S_1^k e_1 \right] \otimes \left[\sum_{k=0}^\infty z^k S_2^k e_2 \right] \right\} + \frac{1}{z} \left\{ e_1 \otimes \left[\sum_{k=1}^\infty z^k S_2^k e_2 \right] \right\} \\
&\quad = \frac{1}{z} \left\{ \left[\sum_{k=0}^\infty z^k S_1^k e_1 \right] \otimes \left[\sum_{k=0}^\infty z^k S_2^k e_2 \right] \right\} - \frac{1}{z} [e_1 \otimes e_2],
\end{aligned}$$

for all $0 < |z| < r$.

Now, it becomes clear that $(S_1 \otimes I) + (P \otimes S_2)$ is a right inverse of $T_1 * T_2$.

Finally, $T_1 * T_2$ is identified with $E_{T_1} \otimes E_{T_2}$, i.e. $E_{T_1 * T_2} = E_{T_1} \otimes E_{T_2}$ is a trivial consequence of Step 1. □

2.2. Partial transformation induced by geometric tensor product. Fix $S \in B_1(\Omega_1)$ and consider the transformation

$$T \to S * T$$

($B_n(\Omega_2)$ to $B_n(\Omega_1 \cap \Omega_2)$). We shall prove that if S is almost the backward unilateral shift then $S * T \sim S \otimes \mathrm{id}_{\mathbf{C}^n}$.

The philosophy of this proof is that "we can read $T \in B_n(\Omega)$ in a different way by reading E_T in a different way."

DEFINITION 2.2.1. Let $T \in B_n(\Omega)$, where Ω is a component of $\mathbf{C} - \sigma_e(T)$. A holomorphic frame γ of E_T (defined near $z_0 \in \Omega$) will be said to be normalized at z_0 if $\gamma^*(z_0)\gamma(z) \equiv I_n$ wherever γ is defined.

Notice that if γ is as in the definition above, then $\gamma(z_0) \perp (\partial^k\gamma/\partial z^k)(z_0)$ for all $k \geq 1$.

DEFINITION 2.2.2. If $T \in B_n(\Omega)$ and Ω is a component of $\mathbf{C} - \sigma_e(T)$, then for each $z_0 \in \Omega$, we define the number $R(T, z_0)$ by

$$\max\{r \leq d(z_0, \sigma_e(T))\colon \text{ for each } |z - z_0| < r, \text{ there is no nontrivial subspace of } \ker(T - z) \text{ perpendicular to } \ker(T - z_0)\}.$$

(Notice: there is no nontrivial subspace of $\ker(T - z)$ perpendicular to $\ker(T - z_0)$ $\Leftrightarrow$ for any basis γ_z of $\ker(T-z)$ and basis γ_{z_0} of $\ker(T-z_0)$, we have $\det[\gamma_{z_0}^*\gamma_z] \neq 0$.)

It is easy to see that $R(T, z_0)$ is invariant under unitary equivalence of T and $R(T, z_0) > 0$ always.

LEMMA 2.2.1. *Let $T \in B_n(\Omega)$ and assume Ω is a component of $\mathbf{C} - \sigma_e(T)$. Fix $z_0 \in \Omega$. Then there is a holomorphic frame defined on the entire disk $\{z\colon |z - z_0| < R(T, z_0)\}$ which is normalized at z_0.*

PROOF. Let $\tilde{\gamma}$ be a global holomorphic frame of E_T on Ω; then

$$\gamma(z) = \tilde{\gamma}(z)[\tilde{\gamma}^*(z_0)\tilde{\gamma}(z)]^{-1}[\tilde{\gamma}^*(z_0)\tilde{\gamma}(z_0)]^{1/2}$$

works. □

LEMMA 2.2.2. *Let $0 \in \Omega$, $T \in B_n(\Omega)$ and $0 < r < R(T, 0)$. Let γ be the holomorphic frame of E_T as defined in the lemma above. Then there is an orthonormal basis $\{e_j^{(k)}\colon j \geq 0,\ 1 \leq k \leq n\}$ ($e_j \stackrel{\text{def}}{=} (e_j^{(1)}, e_j^{(2)}, \ldots, e_j^{(n)})$) such that $\gamma(z) = e_0 + \sum_{j=1}^{\infty} e_j B_j(z)$, where the $B_j(z)$'s are $n \times n$ matrices of analytic functions on $|z| < R(T, 0)$. Moreover, if $\|B_j\|_r^2 \stackrel{\text{def}}{=} \sup_{|z| \leq r}[Tr(B_j^*(z)B_j(z))]$, then $\sum_{j=1}^{\infty} \|B_j\|_r^2 < +\infty$.*

PROOF. Notice that $\{(\partial^k\gamma/\partial z^k)(0)\}_{k=0}^{\infty}$ is a linearly independent set spanning H (see [**C-D,1**, §1]). Also $\gamma(0) \perp (\partial^k\gamma/\partial z^k)(0)$ for all $k \geq 1$ and $\gamma(0)$ is an orthonormal set.

We shall show that Gram-Schmidt orthonormalization $\{e_j\}_{j=0}^{\infty}$ of the set $\{(\partial^k\gamma/\partial z^k)(0)\}_{k=0}^{\infty}$ is a right choice of our orthonormal basis.

Let $H_j = \mathrm{span}\{(\partial^k\gamma/\partial z^k)(0)\colon 0 \leq k \leq j\}$ for each $j \geq 0$.

Let $e_0 = \gamma(0)$ and, for each $j \geq 1$, let $e_j = (e_j^{(1)}, \ldots, e_j^{(n)})$ be an orthonormal basis of $H_j \ominus H_{j-1}$.

Thus $\{e_j^{(l)}: 1 \leq l \leq n,\ j = 0, 1, 2, \ldots\}$ is an orthonormal basis of H and for each $k \geq 1$ we have

$$\frac{1}{k!}\left(\frac{\partial^k}{\partial z^k}\gamma\right)(0) = \sum_{i=1}^{k} e_i A_{ik},$$

where A_{ik} is an $n \times n$ constant matrix.

Now $\gamma(z) = \sum_{j=1}^{\infty}(\sum_{i=1}^{j} e_i A_{ij})z^j + e_0$ on $|z| < R(T, 0)$ and $\sum_{j=1}^{\infty} A_{ij} z^j = e_i^* \gamma(z)$ is convergent in $M(n, \mathbf{C})$.

Let $\sum_{j=1}^{\infty} A_{ij} z^j = z^i \tilde{A}_i(z) = B_i(z)$. Then

$$\gamma(z) = e_0 + \sum_{i=1}^{\infty} e_i B_i(z) \quad \text{for } |z| < R(T, 0).$$

For $A \in M(n, \mathbf{C})$, write $||A||^2 = \operatorname{tr}(A^* A)$.

Assume $0 < r < \delta < R(T, 0)$ and $||\gamma(z)|| = \sum_{j=0}^{\infty} ||z^i \tilde{A}_i(z)||^2 \leq M$ for $|z| \leq \delta$.

By the maximum principle, on $|z| \leq \delta$, we have

$$||\delta^i \tilde{A}_i(z)||^2 \leq \sup_{|\varsigma| = \delta} ||\varsigma^i \tilde{A}_i(\varsigma)||^2 \leq \sup_{|\varsigma| = \delta} ||\gamma(\varsigma)||^2 \leq M.$$

Thus

$$\sum_{i=1}^{\infty} ||B_i||_r^2 = \sum_{i=1}^{\infty} \sup_{|z| \leq r} ||z^i \tilde{A}_i(z)||^2 \leq \sum_{i=1}^{\infty} \sup_{|z| \leq r} ||\varsigma^i \tilde{A}_i(z)||^2 \left(\frac{r}{\delta}\right)^{2i}$$
$$\leq M \sum_{i=1}^{\infty} \left(\frac{r}{\delta}\right)^{2i} < +\infty. \quad \square$$

Our next lemma is a generalization of Lemma 1.1.1.

If $A \in H^\infty \otimes M(n, \mathbf{C})$, let $||A||_\infty^2 \overset{\text{def}}{=} ||\operatorname{tr} A^* A||_\infty$. It is easy to see $||T_A^*|| \leq ||A||_\infty$, where T_A is the matrix analytic Toeplitz operator acting on $H^2 \otimes \mathbf{C}^n = \tilde{H}$ (as row vectors).

Now, if $\{F_k\}_{k=1}^{\infty}$ is a sequence of $n \times n$ matrices, each of them having all entries in H^∞, with $\sum_{k=1}^{\infty} ||F_k||_\infty^2 < \infty$, then $\phi\colon \tilde{H} \to \tilde{H} \oplus \tilde{H} \oplus \cdots$ defined by

$$f \to f \oplus T_{F_1}^* f \oplus T_{F_2}^* f \oplus \cdots$$

is clearly a bounded linear and bounded below mapping; denote its range by R.

LEMMA 2.2.3. *If $g \in H^\infty \cap QC$, let $S_R(g)$ denote $(T_{g\otimes I_n}^* \oplus T_{g\otimes I_n}^* \oplus \cdots)|_R$. Then $S_R(g) \cong (T_{g\otimes I_n}^* + K) \sim T_{g\otimes I_n}^*$, where K is a compact operator.*

PROOF. The operator $S_R(g)$ is similar to $T_{g\otimes I_n}^*$ via ϕ. By [**BDF**], it suffices to show $S_R(g)$ is essentially normal. Let P be the orthogonal projection of $\tilde{H} \oplus \tilde{H} \oplus \cdots$ onto R, then

$$\begin{aligned}(S_R(g))^*(\phi(f)) &= P\{T_{g\otimes I_n}(f) \oplus (T_{g\otimes I_n} \circ T_{F_1}^*(f)) \oplus (T_{g\otimes I_n}^* \circ T_{F_2}^*(f)) \oplus \cdots\} \\ &= \phi(T_{g\otimes I_n}(f)) + P(0 \oplus L_1(f) \oplus L_2(f) \oplus \cdots),\end{aligned}$$

where $L_j = [T_{g\otimes I_n}, T_{F_j}^*]$ which is compact.

Notice that $||L_j|| \leq 2||g||_\infty \circ ||F_j||_\infty$, $j \geq 1$.

Denote the bounded linear mapping $f \to 0 \oplus L_1(f) \oplus L_2(f) \oplus \cdots$ by $0 \oplus L_1 \oplus L_2 \oplus \cdots$ ($\tilde{H}$ into $\tilde{H} \oplus \tilde{H} \oplus \cdots$).

Set $K_\infty = P \circ (0 \oplus L_1 \oplus L_2 \oplus \cdots) \circ \phi^{-1}$ and set

$$K_j = P \circ (0 \oplus L_1 \oplus \cdots \oplus L_{j-1} \oplus L_j \oplus 0 \oplus 0 \oplus \cdots) \circ \phi^{-1}.$$

Then K_j is compact and $K_j \to K_\infty$ follows from

$$||(K_\infty - K_j)\phi(f)||^2 \leq \sum_{k=j+1}^{\infty} ||L_k(f)||_\infty^2 \leq \left\{ 4||g||_\infty^2 \left(\sum_{k=j+1}^{\infty} ||F_k||_\infty^2 \right) \right\} \circ ||f||^2.$$

Therefore K_∞ is compact.

Finally, using $(S_R(g))^*(\phi(f)) = \phi(T_{g\otimes I_n}(f)) + K_\infty(\phi(f))$, we can directly check

$$[S_R(g), (S_R(g))^*] = \phi \circ [T^*_{g\otimes I_n}, T_{g\otimes I_n}] \circ \phi^{-1} + [S_R(g), K_\infty]. \quad \square$$

Now, we are ready to prove the main result of this section.

THEOREM 2.2.4. *For any $T \in B_n(\Omega)$ (where Ω is a component of $\mathbf{C} - \sigma_e(T)$) and $z_0 \in \Omega$, let $0 < r < R(T, z_0)$ and $g(z) = z_0 + rz$. Then*

$$T_g^* * T \cong (T^*_{g\otimes I_n} + K) \sim T^*_{g\otimes I_n},$$

where K is a compact operator.

PROOF. Without loss of generality, we assume $z_0 = 0 \in \Omega$. Let $\gamma(z)$, $\{B_j(z)\}_{j=1}^\infty$ be as in Lemma 2.2.2, and write $\mathcal{B}_j(z) = \overline{B_j(\bar{z})}$. Then by Lemma 2.2.2, the linear mapping

$$\phi\colon f \to f \oplus T^*_{\mathcal{B}_1 \circ g}(f) \oplus T^*_{\mathcal{B}_2 \circ g}(f) \oplus \cdots$$

is bounded and clearly bounded below. Let $R_T = \operatorname{range}(\phi)$ and

$$S_{R_T}(g) = (T^*_{g\otimes I_n} \oplus T^*_{g\otimes I_n} \oplus \cdots)|_{R_T}.$$

By Lemma 2.2.3, the theorem will be true once we show $S_{R_T}(g) \cong T_g^* * T$. This last equivalence is achieved by the holomorphic isometric bundle map from $E_{S_{R_T}}(g)$ to $E_{T_g^* * T}$ (both over the disk $r\mathcal{D}$)

$$[k_{\mathcal{G}(z)} \otimes I_n] \oplus B_1(z)k_{\mathcal{G}(z)} \oplus B_2(z)k_{\mathcal{G}(z)} \oplus \cdots \to k_{\mathcal{G}(z)} \otimes \gamma(z)$$

where $\mathcal{G}(z) = \overline{g^{-1}(\bar{z})}$ and $k_z(\xi) = 1/(1 - \xi z)$ is the reproducing kernel of the Hardy space. $\square$

COROLLARY 2.2.5. *If $z_0 = 0$ and T, Ω, g, r are all as above, then $||T_g^* * T|| = ||T_g^*|| = r$.*

PROOF. By Definition 2.2.1, $||T_g^* * T|| \leq ||T^*|| = r$; also by Weyl's theorem about the spectrum of a compact perturbation (see [**H**]),

$$||T^*_{g\otimes I_n} + K|| = r||T_{z\otimes I_n} + K/r|| \geq r. \quad \square$$

COROLLARY 2.2.6. *If $z_0 = 0$ and $T \in B_n(\Omega)$, $R(T, 0) > 1$, then*

$$T_z^* * T \cong (T^*_{z\otimes I_n} + K) \sim T^*_{z\otimes I_n},$$

where K is a compact operator.

DEFINITION 2.2.3. Let J be the set of Cowen-Douglas operators T with $0 \in \mathbf{C} - \sigma_e(T)$ and $R(T,0) > 1$. Then we define the transformation $\psi\colon J \to \bigcup_{n=1}^{\infty} B_n(\mathcal{D})$ by $\psi(T) = T_z^* * T$.

What is this transformation good for? In addition to Corollaries 2.2.5, 2.2.6, we have "T_1 and T_2 are equivalent to order $k \Leftrightarrow \psi(T_1)$ and $\psi(T_2)$ are equivalent to order k (in particular, $T_1 \cong T_2 \Leftrightarrow \psi(T_1) \cong \psi(T_2)$)," "$T$ and $\psi(T)$ have the same reducibility." Some other properties of this transformation were discussed in [**L**].

Before we studied the transformation, we had the Fourier and Laplace transformations in mind. But it turns out the flavor is quite different. Further modification of the transformation or a new way of studying this transformation is expected.

2.3. The square transformation induced by geometric tensor product.

DEFINITION 2.3.1. We define the square transformation $\Gamma\colon B_1(\Omega) \to B_1(\Omega)$ by $\Gamma(T) = T * T$.

THEOREM 2.3.1. *If $T_1, T_2 \in B_1(\Omega)$ and $T_1 \sim T_2$, then $\Gamma(T_1) \sim \Gamma(T_2)$.*

PROOF. Let $T_2 = Q^{-1} T_1 Q$, where Q is an invertible operator. Then $T_2 \otimes I = (Q^{-1} \otimes Q^{-1})(T_1 \otimes I)(Q \otimes Q)$.

Notice that

$$H(T_2) * H(T_2) = (Q^{-1} \otimes Q^{-1})[H(T_1) * H(T_1)].$$

Now it is easy to check that the following diagram commutes:

$$\begin{array}{ccc} H(T_2) * H(T_2) & \xrightarrow{T_2 \otimes I} & H(T_2) * H(T_2) \\ {\scriptstyle Q^{-1}\otimes Q^{-1}}\big\uparrow & & {\scriptstyle Q^{-1}\otimes Q^{-1}}\big\uparrow \\ H(T_1) * H(T_1) & \xrightarrow{T_1 \otimes I} & H(T_1) * H(T_1). \end{array} \qquad \square$$

Since the curvature of T is exactly one half of the curvature of $\Gamma(T)$, so we have

THEOREM 2.3.2. *If $T_1, T_2 \in B_1(\Omega)$, then $T_1 \cong T_2 \Leftrightarrow \Gamma(T_1) \cong \Gamma(T_2)$.*

THEOREM 2.3.3. *If $g \in H^\infty$ and $T_g^* \in B_1(\Omega)$, then $\Gamma(T_g^*) \cong B_g^*$, the corresponding Toeplitz operator on the Bergman space.*

PROOF. From Lemma 1.2.1, we know $g\colon g^{-1}(\bar{\Omega}) \to \bar{\Omega}$ is a conformal equivalence. For B_g^*, if we repeat the proof of Lemma 1.2.1 word by word, it is easy to see $B_g^* \in B_1(\Omega)$.

Let $\beta_z(\xi) = \pi^{-1}(1 - \xi z)^{-2}$ be the Bergman kernel. Recall $B_g^*(\beta_z) = \overline{g(\bar{z})}\beta_z$, so the mapping

$$k_{\overline{g^{-1}(\bar{z})}} \otimes k_{\overline{g^{-1}(\bar{z})}} \to \beta_{\overline{g^{-1}(\bar{z})}} \qquad (z \in \Omega)$$

gives a holomorphic isometric bundle map from $E_{T_g^* * T_g^*}$ to $E_{B_g^*}$. $\square$

It is well known that for each $\psi \in C(\operatorname{clos}(\mathcal{D}))$, $T_{\phi|_{S^1}}$ and B_ϕ are unitarily equivalent up to compact perturbation (see [**CO**]). We pose a problem here (related to Theorem 2.3.3):

Let $\psi \in C(S^1)$ with harmonic extension $\hat{\phi}$ to $\mathcal{D}$. If $T_\phi \in B_1(\Omega)$, do we have $\Gamma(T_\phi) \cong B_{\hat{\phi}}$?

We hope this approach can lead to a rediscovery of the fact: B_z^* is not unitarily equivalent to any Toeplitz operator on Hardy space.

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